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On the reflective function of polynomial differential system

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Abstract

This article deals with the reflective function of the polynomial differential systems. The results are applied to discussion of the existence of periodic solutions of these systems.

Keywords: Reflective function; Polynomial differential system; Periodic solution

1. Introduction

As we know, in the objective world the motion equation of many objects can be described by the differential system

$$x' = X(t, x). \tag{1}$$

Thus to study the law of motion of these objects is only to discuss the properties of solutions of (1). If $X(t + 2\omega, x) = X(t, x)$ (ω is a positive constant), to study the solutions' behavior of (1) we could use, as introduced in [1], Poincaré mapping. But it is very difficult to seek Poincaré mapping for many systems which are not integrable in finite terms. In 1980's the Russian mathematician Mironenko [2] first established the theory of reflective function (RF). Since then a quite new method to study (1) has been set. If $F(t, x)$ is RF of (1), then its Poincaré mapping can be expressed by [1,2]: $T(x) = F(-\omega, x)$. So now we only need to seek RF.

In the present section, we introduce the concept of the reflective function, which will be used throughout the rest of this article.

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Now we suppose that $X(t,x)$ is a continuously differentiable vector function on $\mathbf{R}\times\mathbf{R^n}$, and that there exists a unique solution for the initial value problem of (1), $x=(x_1, x_2,\dots,x_n)^{\mathrm T}\in\mathbf{R^n}$.

Definition [2]. A continuously differentiable vector function $F(t,x)$ on $\mathbf{R}\times\mathbf{R^n}$ is called RF, if it is a solution of the Cauchy problem

$$F_t(t,x)+F_x(t,x)X(t,x)+X\bigl(-t,F(t,x)\bigr)=0,\qquad F(0,x)=x. \tag{2}$$

In literature [2], it is introduced that if $x=\varphi(t,t_0,x_0)$ is the solution of (1) such that $x(t_0)=x_0$, then its RF can be defined by $F(t,x)=\phi(-t,t,x)$. Thus for any solution $x(t)$ of (1), we have $F(t,x(t))\equiv x(-t)$. Besides, if $F(t,x)$ is RF of (1), then it is also the RF of system $\dot x=X(t,x)+F_x^{-1}R(t,x)-R(-t,F(t,x))$ (here $R(t,x)$ is an arbitrary continuously differentiable function on $\mathbf{R}\times\mathbf{R^n}$). So we can apply the theory of RF to discussing the property of the solutions of such systems. Mironenko [2–6] combined the theory of RF with the integral manifold to discuss the symmetry and other geometric properties of the solutions of (1), and obtained a lot of excellent new conclusions. Alisevich [7–9], Veresovich [10,11] and Zhou [12–14] got many kinds of special RF, established the necessary and sufficient conditions of existence and stability of periodic solutions, and filled in some gaps in the fields of stability theory of differential equations. How to apply RF more widely to discussing the properties of solutions of (1) is also a very important problem. There will be much work for us to do. In this paper we will answer this question of when the first component of RF of the following differential system

$$\begin{cases}\dot x=a(t,x)+b(t,x)y+c(t,x)y^2=P(t,x,y),\\ \dot y=e(t,x)+f(t,x)y+g(t,x)y^2=Q(t,x,y)\end{cases} \tag{3}$$

does not depend on y, i.e., when it has type $I(t,x)$ and what structure this RF is of? We will establish the necessary and sufficient conditions of the first component of the solutions of (3) being even function, and its solutions periodic. (Where all the coefficients of (3) are continuously differentiable on $\mathbb{R}^2$.)

2. Main results

Without loss of generality, we suppose that $I(t,x)=x$. Otherwise we take the transformation $\xi=I(t,x)$, $\eta=y$.

Let the system (3) have RF: $F(t,x,y)=(F_1(t,x,y),F_2(t,x,y))^{\mathrm T}$.

Lemma 1. *If $F_1(t,x,y)=x$, then $a(0,x)=b(0,x)=c(0,x)\overset{\forall x}{\equiv}0$.*

Proof. If $F_1(t,x,y)=x$, then from (2) follows

$$P(t,x,y)+P(-t,x,\bar y)\equiv 0,\qquad \bar y:=F_2(t,x,y),$$

i.e.,

$$a+\bar a+by+cy^2+\bar b\bar y+\bar c\bar y^2\equiv 0 \tag{4}$$

(where and in the following $a:=a(t,x)$, $\bar a:=a(-t,x),\dots,$etc.).

Setting $t = 0$, we have

$$a\big(0, x(0)\big) + b\big(0, x(0)\big)y(0) + c\big(0, x(0)\big)y^2(0) \overset{\forall x(0), y(0)}{\equiv} 0.$$

From this we deduce

$$a(0, x) \equiv b(0, x) \equiv c(0, x) \overset{\forall x}{\equiv} 0. \tag{5}$$

This finishes the proof. □

In the following, we will suppose that condition (5) is satisfied. The notation "$a(t, x) \neq 0$" means that, in some deleted neighborhood of $t = 0$ and $|t|$ being small enough, $a(t, x)$ is different from zero. "System (1) is 2ω-periodic," which means $X(t + 2\omega, x) = X(t, x)$.

Lemma 2. *If $F_1(t, x, y) = x$, $c(t, x) \equiv 0$ and $b(t, x) \neq 0$, then $F_2(t, x, y) = m(t, x) + n(t, x)y$ (where $m(t, x)$, $n(t, x)$-continuously differentiable functions on $\mathbb{R}^2$).*

This conclusion can be deduced directly from (4).

Lemma 3. *Let $F_1(t, x, y) = x$, and $c(t, x) \neq 0$. If $\lim_{t\to 0} b(t, x)/c(t, x)$ exists, then*

$$\lim_{t\to 0} \frac{a(t, x) + a(-t, x)}{c(t, x)} = 0, \qquad \lim_{t\to 0} \frac{c(t, x)}{c(-t, x)} = -1.$$

Proof. Let $\lim_{t\to 0} b(t, x)/c(t, x)$ exist. Applying (4) we bring out

$$\lim_{t\to 0} \frac{a + \bar{a}}{c} + y(0) \lim_{t\to 0} \left(\frac{b}{c} + \frac{\bar{b}}{c}\right) + y^2(0) \lim_{t\to 0} \left(1 + \frac{\bar{c}}{c}\right) \overset{\forall y(0)}{=} 0.$$

Thus proof is finished. □

Theorem 1. *Let $F_1(t, x, y) = x$, and $c(t, x) \neq 0$. $F_2(t, x, y) = m(t, x) + n(t, x)y$, if one of the following conditions is satisfied.*

(1^0) $\lim_{t\to 0} b(t, x)/c(t, x)$ *exists*;
(2^0) $b_x(t, x)c(t, x) = b(t, x)c_x(t, x)$;
(3^0) $c(t, x) = t^{k_1} S_1(t, x)$, $b(t, x) = t^{k_2} S_2(t, x)$, $k_1 < 2k_2 + 1$, $S_1(0, x) \cdot S_2(0, x) \neq 0$ (*where k_1, k_2—positive constants, $S_1(t, x)$, $S_2(t, x)$, $m(t, x)$, $n(t, x)$—continuously differentiable functions on $\mathbb{R}^2$*).

Proof. Let $F_1(t, x, y) = x$. By (4) is yielded

$$\bar{y}^2 = -\frac{1}{\bar{c}}\big[a + \bar{a} + by + cy^2 + \bar{b}\bar{y}\big]. \tag{6}$$

Putting $R = a + \bar{a}$, $S = R + by + cy^2$.
Differentiating (4) implies

$$S_t + S_x P + S_y Q + (\bar{b}_t + \bar{b}_x P)\bar{y} + (\bar{c}_t + \bar{c}_x P)\bar{y}^2 - (\bar{b} + 2\bar{c}\bar{y})Q(-t, x, \bar{y}) \equiv 0. \tag{7}$$

Computing (6) and (7), we obtain

$$B_1(t,x,y) + B_2(t,x,y)\bar{y} \equiv 0, \tag{8}$$

where

$$\begin{aligned} B_1(t,x,y) &= \bar{c}[S_t + S_x P + S_y Q] - S[\bar{c}_t + \bar{c}_x P] + S[2\bar{c}\bar{f} - \bar{b}\bar{g}] - \bar{b}\bar{c}\bar{e} \\ &= \sum_{i=0}^{4} B_{1i}(t,x)y^i, \\ B_2(t,x,y) &= \bar{c}[\bar{b}_t + \bar{b}_x P] - \bar{b}[\bar{c}_t + \bar{c}_x P] - 2\bar{c}^2\bar{e} + 2S\bar{c}\bar{g} + \bar{b}(\bar{c}\bar{f} - \bar{b}\bar{g}) \\ &= \sum_{j=0}^{2} B_{2j}(t,x)y^j. \end{aligned}$$

(1) If $B_2(t,x,y) \equiv 0$, then $B_1(t,x,y) \equiv 0$, i.e.

$$\begin{cases} bc_t - cb_t = b^2 g - bcf - 2acg + 2c^2 e, \\ \bar{c}[c_t + R_x c - bg + 2cf] = c[\bar{c}_t + R\bar{c}_x + \bar{b}\bar{g} - 2\bar{c}\bar{f}], \\ \bar{c}[R_t + R_x a + be] = R[\bar{c}_t + a\bar{c}_x + \bar{b}\bar{g} - 2\bar{c}\bar{f}] + \bar{b}\bar{c}\bar{e}, \\ \bar{c}c_x = c\bar{c}_x, \\ b_x c - bc_x = -2cg. \end{cases} \tag{9}$$

Next consider the following three cases.

(1^0) If $\lim_{t\to 0} b(t,x)/c(t,x)$ exists, then by Lemma 3 is implied

$$\lim_{t\to 0} \frac{a+\bar{a}}{c} = 0, \qquad \lim_{t\to 0} \frac{c}{\bar{c}} = -1.$$

Putting

$$W(t,x) = \left(\frac{b}{c}\right)^2 + \left(\frac{\bar{b}}{\bar{c}}\right)^2 \frac{\bar{c}}{c} - 4\frac{R}{c},$$

then $W(0,x) = 0$.

Computing (9), we get

$$\frac{\partial W(t,x)}{\partial t} = 2\left(f - \frac{bg}{c}\right)W(t,x) + \left(\bar{a} - \frac{\bar{b}^2}{4\bar{c}}\right)\frac{\partial W(t,x)}{\partial x}.$$

In view of [15], we get $W(t,x) \equiv 0$. So from (4) follows

$$\bar{y} = -\frac{\bar{b}}{2\bar{c}} + \sqrt{-\frac{c}{\bar{c}}}\,\frac{b}{2c} + \sqrt{-\frac{c}{\bar{c}}}\,y = m(t,x) + n(t,x)y.$$

Hence in this case the present theorem is true.

(2^0) If $b_x c = c_x b$, then by the fifth formula of (9) is yielded $g(t,x) \equiv 0$. Now we will prove $\lim_{t\to 0} c/b \neq 0$. If it is not true, then by the first formula of (9) we have

$$\left(\frac{c}{b}\right)_t = -f\left(\frac{c}{b}\right) + 2e\left(\frac{c}{b}\right)^2,$$

from this follows $c/b \equiv 0$, so $c \equiv 0$. This is contradictory. Hence $\lim_{t\to 0} c/b \neq 0$, i.e., $\lim_{t\to 0} b/c$ exists. As in the first case, the present result is true.

(3^0) Now consider the third case. At first we prove $\lim_{t\to 0} c/b \neq 0$. If it is not true, then $0 < k_2 < k_1 < 2k_2 + 1$. Hence from the fifth formula of (9) follows

$$g = \frac{S_2 S_{1x} - S_1 S_{2x}}{2S_1} t^{k_2}.$$

By the first formula of (9) is yielded

$$\begin{aligned}&(S_2 S_{1t} - S_1 S_{2t} + S_1 S_2 f)t + S_1 S_2 (k_1 - k_2)\\ &\quad = \frac{S_2^2}{2S_1}(S_2 S_{1x} - S_1 S_{2x}) t^{2k_2 - k_1 + 1} + a(S_{2x} S_1 - S_{1x} S_2)t + 2S_1^2 t^{k_1 - k_2 + 1} e.\end{aligned}$$

From this we get $S_1(0,x)\cdot S_2(0,x)(k_1 - k_2) = 0$. This is not possible, so $\lim_{t\to 0} c/b \neq 0$, i.e., $\lim_{t\to 0} b/c$ exists. From above, the present result is true.

(2) If $B_2(t,x,y) \neq 0$, then by (8) is yielded $\bar{y} = -B_1(t,x,y)/B_2(t,x,y)$. From this and (4) is deduced

$$-\bar{c}B_1^2(t,x,y) = B_2(t,x,y)\big[\bar{b}B_1(t,x,y) + \big(a + \bar{a} + by + cy^2\big)B_2(t,x,y)\big].$$

From this follows

$$\bar{y} = \sum_{k=0}^{4} C_k(t,x) y^k.$$

Applying (4), we obtain

$$a + \bar{a} + by + cy^2 + \bar{b}\left[\sum_{k=0}^{4} C_k(t,x) y^k\right] + \bar{c}\left[\sum_{k=0}^{4} C_k(t,x) y^k\right]^2 \equiv 0.$$

From this is implied $C_2(t,x) = C_3(t,x) = C_4(t,x) \equiv 0$. Hence

$$\bar{y} = C_0(t,x) + C_1(t,x)y = m(t,x) + n(t,x)y.$$

Summarizing the above the proof is completed. $\square$

Theorem 2. *If $c(t,x) \neq 0$, then the RF of (3) is*

$$F(t,x,y) = \big(x, m(t,x) + n(t,x)y\big)^{\mathrm{T}},$$

if and only if

$$\begin{cases} c_x \bar{c} = c\bar{c}_x, \\ c + \bar{c}\exp\big(2\int_0^t \big(\frac{bg}{c} + \frac{\bar{b}\bar{g}}{\bar{c}} - f - \bar{f}\big)\,dt\big) = 0, \\ \bar{c}b^2 + c\bar{b}^2 - 4c\bar{c}(a + \bar{a}) = 0, \\ n\big(\frac{b}{c}\big)_x - \big(\frac{\bar{b}}{\bar{c}}\big)_x = 2\big(\frac{\bar{g}}{\bar{c}} - \frac{g}{c}n\big), \\ n\big(\frac{b}{c}\big)_t - \big(\frac{\bar{b}}{\bar{c}}\big)_t = n\frac{bf}{c} + \frac{\bar{b}\bar{f}}{\bar{c}} - n\frac{g(b^2 - 2ac)}{c^2} - \frac{\bar{g}(\bar{b}^2 - 2\bar{a}\bar{c})}{\bar{c}^2} - 2ne - 2\bar{e}, \\ n = \exp\int_0^t \big(\frac{bg}{c} + \frac{\bar{b}\bar{g}}{\bar{c}} - f - \bar{f}\big)\,dt, \\ m = \frac{b}{2c}n - \frac{\bar{b}}{2\bar{c}} \text{ and } \lim_{t\to 0} m(t,x) = 0. \end{cases} \tag{10}$$

Besides this, if system (3) *is* 2ω*-periodic, and*

$$\lim_{t\to-\omega}\frac{c(t,x)}{c(-t,x)}=-1,$$
$$\lim_{t\to-\omega}\left[\frac{b(t,x)}{c(t,x)}\left(\frac{c(t,x)}{-c(-t,x)}\right)^{1/2}-\frac{b(-t,x)}{c(-t,x)}\right]=0, \tag{11}$$

then all the solutions defined on $[-\omega,\omega]$ *of* (3) *are* 2ω*-periodic.*

Proof. Let $F(t,x,y)=(x,m(t,x)+n(t,x)y)^{\mathrm{T}}$ be RF of (3), then

$$\begin{cases} P(t,x,y)+P(-t,x,m+ny)\equiv 0,\\ m_t+n_t y+(m_x+n_x y)P(t,x,y)+nQ(t,x,y)+Q(-t,x,m+ny)\equiv 0,\\ m(0,x)=0, \qquad n(0,x)=1. \end{cases}$$

Thus

$$\begin{cases} a+\bar a+\bar b m+\bar c m^2=0, \qquad b+\bar b n+2\bar c mn=0,\\ c+\bar c n^2=0, \qquad m_t+m_x a+ne+\bar e+m\bar f+\bar g m^2=0,\\ n_t+m_x b+n_x a+nf+n\bar f+2\bar g mn=0,\\ n_x b+m_x c+ng+n^2\bar g=0, \qquad n_x c=0,\\ n(0,x)=1, \qquad m(0,x)=0. \end{cases} \tag{12}$$

It is not difficult to prove that condition (12) is equivalent to (10). If condition (11) holds, then $F(-\omega,x,y)=(x,y)^{\mathrm{T}}$. By [1], we complete the proof of the present theorem. □

Example 1. Differential system

$$\begin{cases} \dot x=\alpha_1(t)x+\sin t e^{\sin t}y+\alpha_2(t)x^2+\alpha_3(t)e^{\sin t}xy+\alpha_4(t)e^{2\sin t}y^2,\\ \dot y=\beta_1(t)e^{-\sin t}x+\big(\beta_2(t)-\cos t\big)y+\beta_3(t)e^{-\sin t}x^2+\beta_4(t)xy \end{cases}$$

has RF $F(t,x,y)=(x,e^{2\sin t}y)^{\mathrm{T}}$, where $\alpha_i(t)$, $\beta_i(t)$—continuously differentiable odd functions on $\mathbb{R}$ $(i=\overline{1,4})$.

If $\alpha_i(t+2\pi)=\alpha_i(t)$, $\beta_i(t+2\pi)=\beta_i(t)$, $(i=\overline{1,4})$, then all the solutions defined on $[-\pi,\pi]$ of this system are 2π-periodic.

Theorem 3. *Assume* $c(t,x)\neq 0$ *and condition* (9) *holds and function* $\bar y=F_2(t,x,y)$ *defined by* (4) *satisfies* $F_2(0,x,y)=y$*, then* $F(t,x,y)=(x,F_2(t,x,y))^{\mathrm{T}}$ *is RF of system* (3)*. Besides this, if system* (3) *is* 2ω*-periodic, then all the solutions of* (3) *are* 4ω*-periodic and all the solutions defined on* $[-\omega,\omega]$ *of* (3) *are* 2ω*-periodic if* $F_2(-\omega,x,y)=y$.

Proof. In view of the assumptions of the present theorem, the function $F(t,x,y)=(x,F_2(t,x,y))^{\mathrm{T}}$, in which $\bar y=F_2(t,x,y)$ is defined by (4), satisfies the basic relation (2), so $F(t,x,y)$ is RF of (3). Since the coefficients in (4) are 2ω-periodic, then F_2, and F are 2ω-periodic respect to t. Hence all the solutions of (3) are 4ω-periodic [2]. If $F_2(-\omega,x,y)=y$, then all the solutions of (3) are 2ω-periodic [2]. □

Similarly, we obtain the following conclusion.

Theorem 4. *If $c(t,x) \equiv 0$, $b(t,x) \neq 0$, then $F(t,x,y) = (x, m(t,x) + n(t,x)y)^{\mathrm{T}}$ is RF of* (3)*, if and only if,*

$$\begin{cases} b_x\bar{b} - \bar{b}_x b = b\bar{g} - \bar{b}g, \\ b_t\bar{b} - \bar{b}_t b + b(R_x\bar{b} - \bar{b}_x R) = a(\bar{b}g - b\bar{g}) - b\bar{b}(f + \bar{f}) + 2Rb\bar{g}, \\ R_t\bar{b} - \bar{b}_t R + a(R_x\bar{b} - \bar{b}_x R) = \bar{b}(\bar{b}\bar{e} - be) - R\bar{b}\bar{f} + R^2\bar{g}, \\ m = -\frac{R}{b}, \qquad n = -\frac{b}{b}, \\ \lim_{t\to 0}\frac{b(t,x)}{b(-t,x)} = -1, \qquad \lim_{t\to 0}\frac{R(t,x)}{b(-t,x)} = 0 \end{cases}$$

(*where $R = a + \bar{a}$*)*. Besides this, if system* (3) *is 2ω-periodic and*

$$\lim_{t\to -\omega}\frac{b(t,x)}{b(-t,x)} = -1, \qquad \lim_{t\to -\omega}\frac{R(t,x)}{b(-t,x)} = 0,$$

then all the solutions defined on $[-\omega,\omega]$ of system (3) *are 2ω-periodic.*

Example 2. Differential system

$$\begin{cases} \dot{x} = \sin t e^{\sin t} y + \frac{1}{2}\bigl(\sin^2 t - \sin^2 2t\bigr)x^2 + \alpha_1(t)\sin t e^{\sin t} xy \\ \dot{y} = \alpha_2(t)e^{-\sin t}x + \bigl(\alpha_3(t) - \cos t\bigr)y + \alpha_4(t)x^2 - \frac{3}{2}\bigl(\sin^2 t - \sin^2 2t\bigr)xy \\ \qquad - \alpha_1(t)\sin t e^{\sin t} y^2 \end{cases}$$

has RF

$$F(t,x,y) = \left(x, \frac{1}{1-\alpha_1(t)x}\bigl[\sin t\bigl(1 - 4\cos^2 t\bigr)x^2 + \bigl(1 + \alpha_1(t)x\bigr)e^{\sin t}y\bigr]e^{\sin t}\right)^{\mathrm{T}},$$

where

$$\alpha_1(t) = -\frac{1}{8}\sin 2t + \frac{1}{16}\sin 4t,$$

$\alpha_2(t)$, $\alpha_3(t)$—continuously differentiable odd functions on $\mathbb{R}$,

$$\alpha_4(t) = \frac{1}{2}e^{-\sin t}\bigl[4\bigl(\cos^2 t - 1\bigr)\bigl(\cos t - \alpha_3(t)\sin t\bigr) - 8\sin^2 t\cos t - 2\alpha_2(t)\alpha_1(t)\bigr].$$

If $\alpha_2(t + 2\pi) = \alpha_2(t)$, $\alpha_3(t + 2\pi) = \alpha_3(t)$, then all the solutions defined on $[-\pi,\pi]$ of this system are 2π-periodic.

Now consider system

$$\begin{cases} \dot{x} = a_1(t)x + a_2(t)y + a_3(t)x^2 + a_4(t)xy + a_5(t)y^2, \\ \dot{y} = b_1(t)x + b_2(t)y + b_3(t)x^2 + b_4(t)xy + b_5(t)y^2, \end{cases} \tag{13}$$

where $a_i(t)$, $b_i(t)$ $(i = \overline{1,5})$—continuously differentiable functions on $\mathbb{R}$.

Theorem 5. *Assume*

$$\begin{cases} B_x\overline{B} - \overline{B}_x B = B\overline{G} - \overline{B}G, \\ B_t\overline{B} - \overline{B}_t B + B(R_x\overline{B} - \overline{B}_x R) = A(\overline{B}G - B\overline{G}) - B\overline{B}(F + \overline{F}) + 2RB\overline{G}, \\ R_t\overline{B} - \overline{B}_t R + A(R_x\overline{B} - \overline{B}_x R) = -R\overline{F}\,\overline{B} + R^2\overline{G}, \\ \lim_{t\to 0}\frac{B(t,\varphi)}{B(-t,\varphi)} = -1, \qquad \lim_{t\to 0}\frac{R(t,\varphi)}{B(-t,\varphi)} = 0, \\ A(0,\varphi) = B(0,\varphi) = 0, \qquad B(t,\varphi) \neq 0, \end{cases}$$

where

$$
\begin{aligned}
A &= b_1\cos^2\varphi + (b_2 - a_1)\cos\varphi\sin\varphi - a_2\sin^2\varphi,\\
B &= b_3\cos^3\varphi + (b_4 - a_3)\cos^2\varphi\sin\varphi + (b_5 - a_4)\sin^2\varphi\cos\varphi - a_5\sin^3\varphi,\\
F &= a_1\cos^2\varphi + (a_2 + b_1)\cos\varphi\sin\varphi + b_2\sin^2\varphi,\\
G &= a_3\cos^3\varphi + (a_4 + b_3)\cos^2\varphi\sin\varphi + (a_5 + b_4)\cos\varphi\sin^2\varphi + b_5\sin^3\varphi,\\
R &= A(t,\varphi) + A(-t,\varphi).
\end{aligned}
$$

Then for any solution $x(t)$, $y(t)$, $(t \in R)$ of system (13)*, the relation $\varphi(-t) = \varphi(t)$ holds, where $\varphi(t)$ is polar angle $\varphi(t) = \arctan y(t)/x(t)$. Besides this, if system* (13) *is 2ω-periodic, then all the solutions defined on $[-\omega,\omega]$ of system* (13) *are 2ω-periodic, if and only if,*

$$
\lim_{t\to-\omega}\frac{B(t,\varphi)}{B(-t,\varphi)} = -1, \qquad \lim_{t\to-\omega}\frac{R(t,\varphi)}{B(-t,\varphi)} = 0.
$$

Proof. Put $x = \rho\cos\varphi$, $y = \rho\sin\varphi$, $0 \leqslant \varphi \leqslant 2\pi$, $0 \leqslant \rho < +\infty$, then system (13) becomes

$$
\begin{cases}
\dot\varphi = A(t,\varphi) + B(t,\varphi)\rho,\\
\dot\rho = F(t,\varphi)\rho + G(t,\varphi)\rho^2.
\end{cases}
$$

In view of Theorem 4, this proof is completed. □

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